

Lagrange multiplier and Wess-Zumino variable as large extra dimensions in the torus universe

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Abstract

We study the effect of the topology of universe by gauging the non-relativistic particle model on the torus and 3-torus, using the symplectic formalism of constrained systems and embedding those models on extended phase-spaces. Also, we obtain the generators of the gauge transformations for gauged models. Extracting the corresponding Poisson structure of the existed constraints, we show the effect of the topology on the canonical structure of the phase-spaces of those models and suggest some phenomenology to prove the topology of the universe and probable non-commutative structure of the space. In addition, we show that the number of large extra dimensions in the Phase-spaces of the gauged embeded models are exactly two. Moreover, in the classical form, we talk over MOND theory in order to study the origin of the terms appeared in the gauged theory, which modify the Newton's second law.

1 Introduction

1.1 Why torus?

The torus universe model or the doughnut theory of the universe, is the model which describes the universe as a doughnut, having surface with topology of a two dimensional torus. Historically, the first explanations of the shape of the universe were proposed in the mid 60s, after the discovery of the CMB by Starobinsky and Zeldovic [1].

In experimental point of view, data of cosmos radiation measurements gathered by the satellite COBE, shows small discrepancies in temperature fluctuation. This shows that the universe consists of regions of varying densities. Stenemse and Silk proposed that this paradox, i.e. the isotropic universe with different regional densities, suggests that universe may have a complicated geometric structure [2]. In other words, these fluctuations show that multiply connected universes are possible, and the simplest and the most probable one is a 3-torus [2, 3]. Also, simulations of CMB map and the angular power spectrum of the temperature fluctuations, considering the torus topology, and comparing them with the observations of the COBE satellite in order to obtain the lower limit of universe size, suggest that we live in a small universe with the probable topology of the torus [4, 5, 6, 7, 8].

Another direct observation to detect the topology of the CMB maps is an approach which the topology of the universe and the probability of being torus has been investigated and called "Circles in The Sky" [9, 10, 11, 12, 13].

Moreover, data gathered by WMAP satellite shows more intense CMB across one plane of the universe in comparison with others, which forms a straight line in the universe. Where radiation surpasses its quota for the size of the plane seen, one can say that the universe has overflowed in that direction and creates a plane in other directions. Thereby, the invisible loop of a torus may have been created perpendicular to the direction of the plane. Thus, the analyzed CMB maps from data obtained from WMAP has released some results in favour of a torus form of the universe [1, 14, 15]. Measurements of WMAP shows that the universe is flat with only 0.4% margins of error. On the other hand, flat universes with boundaries or edges are not desired mathematically, and thus, they are excluded from consideration. Although there are some finite compact universe models without boundaries, the torus universe is the only one which is both explains a flat and a finite universe [16].

Theoretically, string theory and also theories considering extra large dimensions suggest that we live in a universe with higher dimensions of

space-time and most of the modern cosmological models are founded on such assumptions. Moreover, the problems of the standard cosmology are avoided, considering higher dimensional spacetime, and also most of the predictions of the inflation cosmology are fulfilled via these approaches [17, 18, 19].

In order to combine topological theories and extra large dimensions universe, it has been shown that cyclic universe models can be acquired in a toroidal spacetime which is embedded in a five-dimensional bulk with large extra dimensions, and the three dimensional space has been shown as a closed ring, moving on the surface of the torus [20, 21].

If we expect that the universe has a topology of a torus, we can construct a gauge theory, using the Lagrangian of a particle on the torus, and quantize such a gauge theory, and extract its gauge transformation relations. Our goal to study the motion of a non-relativistic particle on a torus and gauging that model is to obtain a configuration space with extra dimensions. As we know, studying the motion of a free particle is the most available laboratory, in which we can test whether the torus universe has been existed or not. Making such gauge theories and studying its Hamiltonian spectrum may help us to understand the real topology of the universe. Moreover, with investigating the final obtained phase-space, one can check the commutativity and non-commutativity of the universe. In addition, we can determine the ratio of two diameters of the torus.

Another point of view, in which we can study the constructed classical theory on a torus is the MOND theory, that talks over the corrections added to the Newtonian classical mechanics. In the common Poisson structure, Hamiltonian equations of motions and Newton laws are equivalent [22]. In this article, we construct a classical theory which has an unusual Poisson structure due to its constrained structure. This Poisson structure adds some additional terms to the Hamiltonian and consequently to the equations corresponding Newton's second law, which can be studied via the MOND phenomenological theory.

Our tool to construct a gauge theory which reduces to a particle on the torus after gauge fixing is the symplectic gauge analysis approach, will be discussed later.

1.2 Gauge theories and constraints

As we know, gauge invariance is one of the most significant and practical concepts in theoretical physics. This concept is the cornerstone of the standard model of elementary particles. Gauge invariance is due to the presence of the important physical variables which are independent of

the local reference frames [23]. Whenever a change is applied in an arbitrary reference frame, which makes changes in such variables, the gauge transformation is occurred. Such physical variables are called gauge invariant variables.

Generally, we deal with gauge invariance, or in the other words, local invariance, which produces gauge bosons in fundamental interactions. As a physical law, the existence of (local) gauge symmetry in particle physics is the sign of the presence of interactions [24].

It is very important to know that the quantization of gauge theories entails a particular prudence, because of the presence of gauge symmetry, which exist some nonphysical degrees of freedom, that must be eliminated before and after the quantization is applied [25].

On the other hand, in a gauge theory, the equations of motion are not able to determine the dynamics of the system thoroughly at every moment. Thus, one of the most particular features of a gauge theory is the emergence of arbitrary time dependent functions in general solutions of the equations of motion. The emergence of such time dependent functions is companied with the relations between phase-space coordinates, which are called constraints [26].

In order to quantize such systems, the identities between phase-space coordinates are classified into two main groups, by Dirac [27]. The first group are identities which are present in the phase-space, similar to a coordinate or a momentum variable. These identities, which transform the physical system without any changes in the phase-space, are called first-class constraints, and according to Dirac's guess are generators of the gauge transformations in the phase-space. The second group are not related to any degrees of freedom and must be removed. Presence of such identities, which are called second-class constraints, indicates the absence of the gauge symmetry in the system. Therefore, to gauge a system, containing second-class constraints, we must transform them to first-class ones, as a first step [28, 29].

There are some approaches to perform such a conversion, like BFT method [30, 31, 32, 33, 34], the symplectic formalism [25, 35, 36, 37], and the Noether dualization technique [38, 39, 40]. As we mentioned before, in order to gauge a system with second-class constraints, we use the symplectic approach in order to embed a non-invariant system in an extended phase-space [41, 42, 43].

1.3 Symplectic Formalism

Symplectic formalism was introduced by Faddeev and Jackiw [35], to avoid consistency problems which spoil the Poisson brackets algebra

and consequently fail any quantization techniques in constrained systems [44, 45]. The mathematics of this formalism is based on the symplectic structure of the phase-space, and therefore, is different from other approaches. Also, in the symplectic formalism there is no distinction between the first and second-class constraints as in the case of the other quantization procedures [37].

The starting point of the symplectic approach is a Lagrangian which is first order in the time derivatives. All second order Lagrangian terms can be converted to first order ones by enlarging the corresponding configuration space so that it includes the conjugate momentum of the coordinate variables [46]. Being dependent only on first order Lagrangian makes the symplectic approach independent from the classification of the constraints into primary, secondary, etc. [49]. In this approach, instead of solving the constraints, one adds their time derivatives to the Lagrangian and considers the corresponding Lagrange multipliers as additional coordinates [47]. Also, to convert the nature of second-class constraints to first ones, the phase-space would be extended with the help of Wess-Zumino variables [50]. After such a conversion, choosing conventional zero-modes which are the generators of the gauge transformations and obey particular boundary conditions, one can eliminate Wess-Zumino variables, which makes the gauged model equivalent to the original system [51].

2 Gauging a non-relativistic particle model on the torus

2.1 Particle on the torus

In the first part of this article we assume a non-relativistic particle on a torus in a three-dimensional configuration space. Considering this model, the particle lives on a two-dimensional configuration space, effectively. After our gauging process, we will see that at least one dimension is added to the previous configuration space, which makes its space more realistic.

In all sections of this article we get the radiuses of the torus 1 and ς , in order to use dimensionless coordinates. Thus, in spherical coordinates, the surface of a torus is defined by

$$\begin{aligned} x &= (1 + \varsigma \cos \theta) \cos \varphi \\ y &= (1 + \varsigma \cos \theta) \sin \varphi \\ z &= \varsigma \sin \theta \end{aligned} \tag{1}$$

The surface of the torus is described by primary constraint $\phi_1 = 0$ in configuration space for free particle on it.

$$\phi_1(r, \theta) = r^2 - 2\varsigma \cos \theta - (1 + \varsigma^2). \quad (2)$$

In this coordinate canonical Hamiltonian for unit mass is

$$H_c = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta}). \quad (3)$$

In formal constrained analysis we arrive to secondary (finally) constraint in Phase-space as

$$\phi_2(r, \theta, p_r, p_\theta) = 2(rp_r + \frac{\varsigma p_\theta \sin \theta}{r^2}). \quad (4)$$

The set of constraints form a second-class system with non-constant Δ matrix as

$$\Delta_{12} = 4(r^2 + \frac{\varsigma^2 \sin^2 \theta}{r^2}), \quad (5)$$

which makes its embedding by BFT method problematic. This is the reason that we use the symplectic approach which is not affected by the Poisson structure of second-class constraints.

2.2 Symplectic analysis of a particle on the torus

Constructing first-class models from a singular Lagrangian is more straightforward in the symplectic formalism than other similar approaches. This is done by embedding the primary model in an extended phase-space.

In this model, the singularity nature of the free particle Lagrangian due to its configuration constraint, $\phi_1(r, \theta)$, can be imposed by a new dynamical variable (say undetermined Lagrange multiplier) λ , in such a way that adds the constraints to the free Lagrangian,

$$L^{(0)} = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - H_c - \lambda_1\phi_1(r, \theta). \quad (6)$$

Note; In this article the Greek indices, α , β , $\tilde{\alpha}$, and $\tilde{\beta}$, are used to determine the phase-space variables.

The symplectic variables and their conjugate momenta as the symplectic one-form can be read off from the model straightforwardly,

$$\begin{aligned} \xi_\alpha^{(0)} &= (r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda), \\ \mathcal{A}_\alpha^{(0)} &= (p_r, p_\theta, p_\varphi, 0, 0, 0, 0). \end{aligned} \quad (7)$$

The symplectic two-form is defined by

$$f_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta^{(0)} - \partial_\beta \mathcal{A}_\alpha^{(0)}. \quad (8)$$

Thus, the zeroth-iterated symplectic two-form, using the equation (8) and the symplectic variables and the corresponding conjugate momenta (7), will be obtained as follows:

$$f_{\alpha\beta}^{(0)} = \begin{pmatrix} \mathbf{0}_{3 \times 3} & -\mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 1} \\ \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} & 0 \end{pmatrix}. \quad (9)$$

This matrix is singular, and so, it has the following null vector,

$$n_\alpha^{(0)} = (\mathbf{0}_{1 \times 3} \quad \mathbf{0}_{1 \times 3} \quad 1). \quad (10)$$

Using the zero iterative potential,

$$\mathcal{V}^{(0)} = H_c + \lambda_1 \phi_1, \quad (11)$$

the first constraint (2) will be obtained from following formula.

$$\phi_1 = n_\alpha^{(0)} \frac{\partial \mathcal{V}^{(0)}}{\partial \xi^{(0)\alpha}}. \quad (12)$$

Substituting the first constraint, obtained from (12) into the original Lagrangian, we can put the constraint into the kinetic part of the Lagrangian. It means that we make the primary constraint ϕ_1 as a momentum conjugate to the variable λ_1 . In other words, we convert the strongly nonlinear constraint, ϕ_1 , into the momentum (linear constraint) of the phase-space. Hence, the first iterative Lagrangian will be obtained as

$$L^{(1)} = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - \dot{\lambda}_1\phi_1 - H_c. \quad (13)$$

We see that the constraint is omitted from the potential. So, for the first iterative potential we have,

$$\mathcal{V}^{(1)} = H_c. \quad (14)$$

Now, we read off new symplectic variables and one-form from (13),

$$\begin{aligned} \xi_\alpha^{(1)} &= (r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1), \\ \mathcal{A}_\alpha^{(1)} &= (p_r, p_\theta, p_\varphi, 0, 0, 0, \phi_1). \end{aligned} \quad (15)$$

The corresponding symplectic two-form is constructed as,

$$f_{\alpha\beta}^{(1)} = \begin{pmatrix} \mathbf{0}_{3 \times 3} & -\mathbf{1}_{3 \times 3} & \mathbf{u}_{1 \times 3}^T \\ \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 1} \\ -\mathbf{u}_{1 \times 3} & \mathbf{0}_{1 \times 3} & 1 \end{pmatrix}, \quad (16)$$

which;

$$u_\alpha = (2r \quad 2\varsigma \sin \theta \quad 0) \quad (17)$$

The two-form (16) is a singular one and it has following null vectors,

$$\begin{aligned} n_{1\alpha}^{(1)} &= (\mathbf{0}_{1 \times 3} \quad \mathbf{u}_{1 \times 3} \quad 0), \\ n_{2\alpha}^{(1)} &= (\mathbf{0}_{1 \times 3} \quad \mathbf{0}_{1 \times 3} \quad 1). \end{aligned} \quad (18)$$

From linear algebra, we know that the linear combination of these null vectors is also a null vector,

$$n_\alpha = n_{1\alpha}^{(1)} + h n_{2\alpha}^{(1)} \quad (19)$$

Using (12), we obtain the second constraint.

$$\phi_2 = 2(rp_r + \frac{\varsigma p_\theta \sin \theta}{r^2}). \quad (20)$$

Now, the second iterative Lagrangian is

$$L^{(2)} = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\varphi}p_\varphi - \dot{\lambda}_1\phi_1 - \dot{\lambda}_2\phi_2 - H_c, \quad (21)$$

and new symplectic variables and one-form are

$$\begin{aligned} \xi_\alpha^{(2)} &= (r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1, \lambda_2), \\ \mathcal{A}_\alpha^{(2)} &= (p_r, p_\theta, p_\varphi, 0, 0, 0, \phi_1, \phi_2), \end{aligned} \quad (22)$$

with which we can construct the following symplectic two-form,

$$f_{\alpha\beta}^{(2)} = \begin{pmatrix} \mathbf{0}_{3 \times 3} & -\mathbf{1}_{3 \times 3} & \mathbf{u}_{1 \times 3}^T & \mathbf{v}_{1 \times 3}^T \\ \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 1} & \mathbf{w}_{1 \times 3}^T \\ -\mathbf{u}_{1 \times 3} & \mathbf{0}_{1 \times 3} & 0 & 0 \\ -\mathbf{v}_{1 \times 3} & -\mathbf{w}_{1 \times 3} & 0 & 0 \end{pmatrix}. \quad (23)$$

and, \mathbf{v} and \mathbf{w} are row matrices which are defined as follow,

$$\begin{aligned} v_\alpha &= (2(p_r - \frac{2\varsigma p_\theta \sin \theta}{r^3}) \quad \frac{2\varsigma p_\theta \cos \theta}{r^2} \quad 0), \\ w_\alpha &= (2r \quad \frac{2\varsigma \sin \theta}{r^2} \quad 0). \end{aligned} \quad (24)$$

The corresponding symplectic two-form is non-singular. Thus, it does not have any null vector and consequently the iterative process stops and no other constraint will be obtained.

Now, we start the symplectic embedding procedure to convert second-class constraints to first ones. The main idea of this procedure is to adjoin the Wess-Zumino variables to the original phase-space [50]. In order to do that, we expand the original Phase-space by introduction of a function G as WZ Lagrangian, depending on the original phase-space variables and the WZ variable σ , as the expansion in terms of the WZ variables, defined by

$$G(r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1, \sigma) = \sum_{n=0}^{\infty} \mathcal{G}^{(n)}. \quad (25)$$

This function is gauging potential and satisfies the following boundary condition by vanishing $\mathcal{G}^{(0)}$,

$$G(r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1, \sigma = 0) = 0. \quad (26)$$

Introducing the new term G into the original symmetrized Lagrangian (13), we obtain a Lagrangian which depends on both original coordinates and WZ variables,

$$\begin{aligned} \tilde{L}^{(1)} &= L^{(1)} + L_{WZ}, \\ &= L^{(1)} + G(r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1, \sigma). \end{aligned} \quad (27)$$

By extending the phase-space, symplectic variables and one-form will be extended as

$$\begin{aligned} \tilde{\xi}_{\tilde{\alpha}}^{(1)} &= (r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1, \sigma), \\ \tilde{\mathcal{A}}_{\tilde{\alpha}}^{(1)} &= (p_r, p_\theta, p_\varphi, 0, 0, 0, \phi_1, 0). \end{aligned} \quad (28)$$

Calculating corresponding symplectic two-form we have

$$\tilde{f}_{\tilde{\alpha}\tilde{\beta}}^{(1)} = \begin{pmatrix} f_{\alpha\beta}^{(1)} & \mathbf{0}_{7 \times 1} \\ \mathbf{0}_{1 \times 7} & 0 \end{pmatrix}, \quad (29)$$

which has the following zero modes

$$\begin{aligned} \tilde{n}_{1\tilde{\alpha}}^{(1)} &= \begin{pmatrix} n_{1\alpha}^{(1)} & 1 \end{pmatrix}, \\ \tilde{n}_{2\tilde{\alpha}}^{(1)} &= \begin{pmatrix} n_{2\alpha}^{(1)} & 0 \end{pmatrix}. \end{aligned} \quad (30)$$

These null vectors are the generators of gauge symmetries, since their contraction with the gradient of the potential does not produce any constraint [25]. We can use the linear combination of these null vectors,

$$\tilde{n}_{\tilde{\alpha}} = \tilde{n}_{1\tilde{\alpha}}^{(1)} + \tilde{h}\tilde{n}_{2\tilde{\alpha}}^{(1)} \quad (31)$$

In order to compute L_{WZ} , we must be assured that no other constraint is produced. This mandatory condition generates an iterative system of differential equations, defined by the following equation,

$$\tilde{n}_{\tilde{\alpha}} \frac{\partial \mathcal{V}^{(1)}}{\partial \tilde{\xi}^{(0)\tilde{\alpha}}} = \frac{\partial \mathcal{G}^{(n)}}{\partial \sigma}. \quad (32)$$

Substituting (14) into (32), we determine $\mathcal{G}^{(1)}$ after an integration process as

$$\mathcal{G}^{(1)} = (2rp_r + \frac{2\varsigma p_\theta \sin \theta}{r^2})\sigma. \quad (33)$$

Putting $\mathcal{G}^{(1)}$ into (27), the first iterative Lagrangian will be obtained. Hence, for the first-iterated potential we have,

$$\tilde{\mathcal{V}}^{(1)} = H_c - \mathcal{G}^{(1)}. \quad (34)$$

Using the (32) for the second time to get $\mathcal{G}^{(2)}$, we will have,

$$\mathcal{G}^{(2)} = -2(r^2 + \frac{\varsigma \sin^2 \theta}{r^2})\sigma^2. \quad (35)$$

Substituting $\mathcal{G}^{(2)}$ into the first iterative Lagrangian, we will obtain the second iterative Lagrangian. Consequently, the second-iterated potential is

$$\tilde{\mathcal{V}}^{(1)} = H_c - \mathcal{G}^{(1)} - \mathcal{G}^{(2)}. \quad (36)$$

Again, using (32) to obtain $\mathcal{G}^{(3)}$, we will see that $\frac{\partial \mathcal{G}^{(3)}}{\partial \sigma} = 0$, and so, the zero-mode (31) does not make a new constraint. In conclusion, all correction terms $\mathcal{G}^{(n)}$, with $n \geq 3$ are vanished. Thus, the gauge invariant canonical Hamiltonian, which had been defined as the symplectic potential, is obtained from

$$\begin{aligned} \tilde{H}_{(c)} &= \mathcal{V}^{(1)} + G(r, \theta, \varphi, p_r, p_\theta, p_\varphi, \lambda_1, \sigma), \\ &= H_c + \lambda_1 \phi_1 - \mathcal{G}^{(1)} - \mathcal{G}^{(2)}, \end{aligned} \quad (37)$$

and the gauged Lagrangian (27), will be

$$\tilde{L}^{(1)} = L^{(1)} + \mathcal{G}^{(1)} + \mathcal{G}^{(2)}. \quad (38)$$

The generators of infinitesimal gauge transformations can be obtained, using $\varepsilon_i \phi^i$, which ϕ^i are first-class constraints [52, 53]. Also, substituting zero-modes (30) in the following relation,

$$\delta \tilde{\xi}_{\tilde{\alpha}}^{(1)} = \varepsilon_i \tilde{n}_{\tilde{\alpha}}^{i(1)}, \quad (39)$$

one can obtain the following infinitesimal gauge transformations [44, 49].

$$\begin{aligned} \delta r &= 0, & \delta p_r &= 2r\varepsilon_1, \\ \delta \theta &= 0, & \delta p_\theta &= 2\varepsilon_1 \sin \theta, \\ \delta \varphi &= 0, & \delta p_\varphi &= 0, \\ \delta \lambda &= \varepsilon_2, & \delta \sigma &= \varepsilon_1, \end{aligned} \quad (40)$$

which ε_i are infinitesimal time dependent parameters. Obtaining a non-linear first-order Lagrangian, we have constructed a gauge theory with the corresponding nonlinear generator functions of gauge transformations for the model. Thus, the gauge symmetry of the model is determined via these transformations. In other words, the gained model is invariant under these transformations.

To obtain gauge symmetries of the model, one can use the Poisson brackets of the first-class constraints and symplectic variables via the following relation [52, 53],

$$\delta \tilde{\xi}_{\tilde{\alpha}}^{(1)} = \{\tilde{\xi}_{\tilde{\alpha}}^{(1)}, \phi_j\} \varepsilon_j. \quad (41)$$

Apparently the results obtained from (41) is the same as the infinitesimal gauge transformations (40).

Considering constrained analysis of the Lagrangian (38) and segregating its corresponding constraints in the following section, we study the gauge symmetry of the model more easily.

2.3 Constraint structure of the gauged Lagrangian

Using the symplectic method, we enhance the gauge symmetry of the primary model. In following, we derive constraints and phase-space structure of the gauged Lagrangian (38). In this gauged model, new dynamical variables λ and σ appear first-orderly in the Lagrangian. So, their momenta are primary constraints in the phase-space. Thus,

$$\begin{aligned} \frac{\partial \tilde{L}^{(0)}}{\partial \tilde{\lambda}^{(1)}} &= 0 \rightarrow \rho_1 = p_\lambda, \\ \frac{\partial \tilde{L}^{(0)}}{\partial \tilde{\sigma}^{(2)}} &= 0 \rightarrow \rho_2 = p_\sigma. \end{aligned} \quad (42)$$

So, the total Hamiltonian, corresponding to Lagrangian (38), is

$$\tilde{H}_T = \tilde{H}_c + \omega^i \rho_i. \quad (43)$$

In the chain-by-chain method [47], the consistency of each individual constraint, i.e. ρ_1 and ρ_2 , starts a chain and gives the next element of that chain. Also, the consistency of second-class constraints determines some of Lagrange multipliers, ω^j , while the consistency of first-class ones leads to constraints of the next level,

$$\begin{aligned} 0 &= \{\rho_i, \tilde{H}_T\}, \\ 0 &= \{\rho_i, \tilde{H}_c\} + \omega^j \{\rho_i, \rho_j\}. \end{aligned} \quad (44)$$

We see that primary constraints are Abelian, i.e. $\{\rho_i, \rho_j\} = 0$. So, we arrive to secondary constraints $\psi_i = \{\rho_i, \tilde{H}_c\}$, where $\psi_1 = \phi_1$ and $\psi_2 = \phi_2$.

The consistency of second level of constraints gives no new constraints, Since,

$$\{\psi_1, \tilde{H}_c\} = -\psi_2, \quad \{\psi_1, \tilde{H}_c\} \neq 0. \quad (45)$$

The first relation is identically true on the constrained surface, and the second one determines a Lagrange multiplier due to the fact that $\{\psi_2, \rho_2\} \neq 0$.

All in all, we have the following chain structures,

$$\begin{aligned} \rho_1 &\rightarrow \psi_1 \rightarrow \psi_2 \rightarrow \times \quad , \\ \rho_2 &\rightarrow \psi_2 \rightarrow \times \quad . \end{aligned} \quad (46)$$

Calculating all Poisson brackets, we see that ρ_1 is a first-class constraint. The Poisson bracket matrix of other four constraints is non-singular. Non-vanishing elements of that matrix are

$$\{\rho_2, \psi_2\} = \frac{w_1^2}{4}(4 + w_2^2), \quad (47)$$

$$\{\psi_1, \psi_2\} = -\frac{w_1^2}{4}(4 + w_2^2), \quad (48)$$

in which, w_1 and w_2 are defined as the components of the row matrix (24).

Since the matrix of Poisson brackets is a square matrix with odd dimensions, it is a singular matrix. This singularity shows that we have more than one first-class constraint in our model, other than ρ_1 . Redefining those constraints and requesting first-class conditions, we will obtain one extra first-class constraint, and in conclusion, there will remain just two second-class constraints.

So, the second first-class constraint is the linear combination of ρ_2 and ψ_1 , as $\Phi_3 = \rho_2 + \psi_1$. This first-class constraint strongly commute with the two remained second-class constraints, as same as ρ_1 ,

$$\{\Phi_3, \psi_1\} = \{\Phi_3, \psi_2\} = 0. \quad (49)$$

Therefore, one can rewrite all constraints in the following notation,

$$\begin{aligned} \Phi_1^{(0)} &= p_\lambda, \\ \Phi_1^{(1)} &= \psi_1, \\ \Phi_2^{(1)} &= \psi_2, \\ \Phi_3 &= \rho_2 + \Phi_1^{(1)}. \end{aligned} \quad (50)$$

which $\Phi_1^{(0)}$ and Φ_3 are first-class constraints, and $\Phi_1^{(1)}$ and $\Phi_2^{(1)}$ are second-class ones.

Now, we put all second-class constraints into the Hamiltonian to calculate the corresponding Dirac brackets. Also, we take first-class constraints intact, because they obey the Abelian algebra.

$$\tilde{H}_c = H_c + \lambda_1 \Phi_1^{(1)} + \sigma \Phi_2^{(1)}. \quad (51)$$

By counting the dimensions of new variables in extended phase-space, we find that $[\lambda_1] = (Length)^{-4}$, and $[\sigma] = (Length)^{-2}$. Thus, redefining the following variables with length scales,

$$\lambda_1 = \lambda'^{-4}, \quad \sigma = \sigma'^{-2}, \quad (52)$$

and replacing them in the Hamiltonian (51), we have,

$$\tilde{H}_c = H_c + \frac{1}{\lambda'^4} \Phi_1^{(1)} + \frac{1}{\sigma'^2} \Phi_2^{(1)}. \quad (53)$$

We see that two variables with length dimensions have been added to our phase-space. Hence, these length scales extend our configuration space from three to five. As a matter of fact, λ' and σ' can be interpreted as large extra dimensions, which are added to spatial part of the phase-space, via the potential which carries them in the Hamiltonian. This result, i.e. having two large extra dimensions, is in a good accordance with [48].

2.4 Quantization of the primary model and the gauged model

Taking into the account two primary constraints of the original model, ϕ_1 and ϕ_2 , and two second-class constraints, $\Phi_1^{(1)}$ and $\Phi_2^{(1)}$ of the gauged

model, and calculating their corresponding Poisson brackets matrix, we will have,

$$\Delta_{ij} = \begin{pmatrix} 0 & -\frac{w_1^2}{4}(4 + w_2^2) \\ \frac{w_1^2}{4}(4 + w_2^2) & 0 \end{pmatrix}, \quad (54)$$

which w_i are the components of the row matrix (24).

In order to determine all Dirac brackets of the original and gauged model, we put the inverse of Δ_{ij} , in the following formula,

$$\{\xi_{\bar{\alpha}}, \xi_{\bar{\beta}}\}^* = \{\xi_{\bar{\alpha}}, \xi_{\bar{\beta}}\} - \{\xi_{\bar{\alpha}}, \Phi_i^{(1)}\} \Delta_{ij}^{-1} \{\Phi_j^{(1)}, \xi_{\bar{\beta}}\}. \quad (55)$$

Non-vanishing Dirac brackets, using components of the row matrices (17) and (24), which are common in both primary and gauged models are

Dirac Brackets	Primary Model	Gauged Model
(r, p_r)	$1 - \frac{4}{4+w_2^2}$	$1 - \frac{8}{4+w_2^2}$
(θ, p_θ)	$1 + \frac{w_2^2}{4+w_2^2}$	$1 + \frac{2w_2^2}{4+w_2^2}$
(φ, p_φ)	1	1
(r, p_θ)	$\frac{4u_2}{u_1(4+w_2^2)}$	$\frac{8u_2}{u_1(4+w_2^2)}$
(θ, p_r)	$\frac{4w_2}{u_1(4+w_2^2)}$	$\frac{8w_2}{u_1(4+w_2^2)}$
(p_r, p_θ)	$\frac{4v_2 - u_1 v_1 w_2}{u_1(4+w_2^2)}$	$\frac{u_2[4w_1\sigma(\frac{1}{4} - w_2^2) - v_1 + 2\sigma v_2] - v_2}{\frac{w_2^2}{8}(4+w_2^2)}$
(σ, p_σ)	N/A	1
(λ, p_λ)	N/A	1
(p_r, p_σ)	N/A	$2u_1$
(p_θ, p_σ)	N/A	$-2u_2$

Table 1: Dirac brackets between extended phase-space variables of the torus model

It is evident that we have obtained some non-commutativity in momentum parts of the phase-space in quantization process. This non-commutativity is due to the quantization on the curved space.

Expanding the Poisson brackets of the gauged model with respect to the ratio of radiuses of the torus, and considering the $\varsigma \rightarrow 0$, the effect of

the topology on the Poisson structure of the phase-space can be studied.

$$\begin{aligned}
\{r, p_r\}^* &\approx 1 + O(\zeta^2), \\
\{\theta, p_\theta\}^* &\approx 1 + O(\zeta^2), \\
\{r, p_\theta\}^* &\approx \frac{2\zeta \sin \theta}{r} + O(\zeta^2), \\
\{\theta, p_r\}^* &\approx -\frac{2\zeta \sin \theta}{r^3} + O(\zeta^2), \\
\{p_r, p_\theta\}^* &\approx \frac{2\zeta (-p_\theta \cos \theta - r p_r \sin \theta + 4r^2 \sigma \sin \theta)}{r^3} + O(\zeta^2), \\
\{p_\theta, p_\sigma\}^* &\approx -4\zeta \sin \theta + O(\zeta^2).
\end{aligned} \tag{56}$$

As we see, these Dirac brackets do not have the common canonical structure. This deformation from canonical structure is due to the change in the topology of the configuration space from \mathbb{R}^n to \mathbb{T}_g .

If our test particle can feel such a deviation, we can do some phenomenological research, i.e. we can compare the deviations in the right-hand side of the equations (56) with the discrepancies obtained from cosmological data, like the spectrum of energy level of the free particle to check whether it is quantized or it is continuous, in order to determine the topology of the universe.

Also, by characterizing first-class constraints and Dirac brackets of a classical system, its quantized model, say Hilbert space of the quantum states, is fully available at tree level, according to Dirac prescription,

$$\{A, B\}^* \rightarrow \frac{1}{i\hbar} [A, B], \quad \hat{\phi}_{FC} | phys \rangle = 0, \tag{57}$$

which $\hat{\phi}_{FC}$ is a quantized version of the first-class constraint. Thus, due to (56) in quantized model we derive a non-commutative structure in the momentum part of the phase-space. Hence, the quantum mechanics obtained from these commutators, is another laboratory to investigate the truth of the relations (56), and in conclusion, the hypothesis which says that we may live on a torus. Also, we can extract some phenomenology from these equations to check the non-commutativity structure of the space.

3 Gauging a non-relativistic particle model on the 3-torus

3.1 Constraints of a particle on the 3-torus

As we have mentioned before, a scenario for the universe as a whole is the boundary of a four-dimensional 3-torus. In this section, we consider

a test particle on a 3-torus and repeat the previous calculations to derive a gauged model for phenomenological purposes.

The surface of the 3-torus describes the primary constraint $\bar{\phi}_1 = 0$ for a particle living on it.

$$\bar{\phi}_1(x, y, z, s) = x^2 + \left(\sqrt{y^2 + \left(\sqrt{z^2 + s^2} - \bar{\zeta}_2 \right)^2} - \bar{\zeta}_1 \right)^2 - 1. \quad (58)$$

This 3-torus is described by three dimensionless radiuses as 1, $\bar{\zeta}_1$, and $\bar{\zeta}_2$. One can reduce this constraint to a polynomial by eliminating the radicals (See A).

The corresponding canonical Hamiltonian for unit mass is

$$\bar{H}_c = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2 + p_s^2). \quad (59)$$

3.2 Symplectic analysis of a particle on the 3-Torus

Here, like the torus model, we have a polynomial which its consistency will determine Lagrange multiplier. Hence, the constrained structure of this model is similar to the previous one, and the only difference will be occurred in the explicit form of the constraints. Thus, the corresponding gauging process for these two models are similar, but because of the greater configuration space for 3-torus, we will obtain a new Poisson structure.

The free particle Lagrangian is singular, due to its configuration space, affected by a new dynamical variable as an undetermined Lagrange multiplier, $\bar{\lambda}$, which adds a constraint to the free Lagrangian.

$$\bar{L}^{(0)} = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z + \dot{s}p_s - \bar{H}_c - \bar{\lambda}_1 \bar{\phi}_1(x, y, z, s). \quad (60)$$

The symplectic variables and symplectic one-form can be read off from the Lagrangian,

$$\begin{aligned} \bar{\xi}_\alpha^{(0)} &= (x, y, z, s, p_x, p_y, p_z, p_s, \bar{\lambda}), \\ \bar{\mathcal{A}}_\alpha^{(0)} &= (p_x, p_y, p_z, p_s, 0, 0, 0, 0, 0). \end{aligned} \quad (61)$$

Starting the symplectic procedure, the corresponding symplectic two-form is obtained as,

$$\bar{f}_{\alpha\beta}^{(0)} = \begin{pmatrix} \mathbf{0}_{4 \times 4} & -\mathbf{1}_{4 \times 4} & \mathbf{0}_{4 \times 1} \\ \mathbf{1}_{4 \times 4} & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 1} \\ \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & 0 \end{pmatrix}. \quad (62)$$

This matrix is singular and so, it has the following null vector,

$$\bar{n}_\alpha^{(0)} = \begin{pmatrix} \mathbf{0}_{1 \times 4} & \mathbf{0}_{1 \times 4} & 1 \end{pmatrix}. \quad (63)$$

Using the zero iterative potential,

$$\bar{V}^{(0)} = H_c + \bar{\lambda}_1 \bar{\phi}_1, \quad (64)$$

the first constraint will be obtained from following formula.

$$\bar{\phi}_1 = \bar{n}_\alpha^{(0)} \frac{\partial \bar{V}^{(0)}}{\partial \bar{\xi}^{(0)}_\alpha}, \quad (65)$$

which gives the (58).

In order to remove the constraint from Hamiltonian and add it to the kinetic part of the Lagrangian, we substitute (65) in the Lagrangian (60). As a result, the first iterative Lagrangian will be obtained as,

$$\bar{L}^{(1)} = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z + \dot{s}p_s - \dot{\bar{\lambda}}_1 \bar{\phi}_1 - \bar{H}_c, \quad (66)$$

and the first iterative potential will be,

$$\bar{V}^{(1)} = \bar{H}_c. \quad (67)$$

Then, new symplectic variables and one-form are

$$\begin{aligned} \bar{\xi}_\alpha^{(1)} &= (x, y, z, s, p_x, p_y, p_z, p_s, \bar{\lambda}), \\ \bar{\mathcal{A}}_\alpha^{(1)} &= (p_x, p_y, p_z, p_s, 0, 0, 0, 0, \bar{\phi}_1), \end{aligned} \quad (68)$$

which gives the corresponding symplectic two-form,

$$\bar{f}_{\alpha\beta}^{(1)} = \begin{pmatrix} \mathbf{0}_{4 \times 4} & -\mathbf{1}_{4 \times 4} & \mathcal{U}_{1 \times 4}^T \\ \mathbf{1}_{4 \times 4} & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 1} \\ -\mathcal{U}_{1 \times 4} & \mathbf{0}_{1 \times 4} & 1 \end{pmatrix}, \quad (69)$$

and,

$$\mathcal{U}_\mu = \frac{\partial \bar{\phi}_1}{\partial \bar{q}^\mu}. \quad (70)$$

which \bar{q}^μ is the spatial component of the symplectic phase-space, i.e. x, y, z, s . Since, the tensor (69) is a singular one, it has following null vectors,

$$\begin{aligned} \bar{n}_{1\alpha}^{(1)} &= (\mathbf{0}_{1 \times 4} \quad \mathcal{U}_\mu \quad 0), \\ \bar{n}_{2\alpha}^{(1)} &= (\mathbf{0}_{1 \times 4} \quad \mathbf{0}_{1 \times 4} \quad 1). \end{aligned} \quad (71)$$

The linear combination of these null vectors is also a null vector for (69).

$$\bar{n}_\alpha = \bar{n}_{1\alpha}^{(1)} + \bar{h}\bar{n}_{2\alpha}^{(1)}. \quad (72)$$

Using (58), we find the second constraint,

$$\bar{\phi}_2 = \bar{\mathcal{A}}_\alpha^{(0)} \partial^\alpha \bar{\phi}_1, \quad (73)$$

which ∂^α is the derivative with respect to the symplectic variables $\xi^{(0)\alpha}$.

Now, the second iterative Lagrangian will be

$$\bar{L}^{(2)} = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z + \dot{s}p_s - \dot{\bar{\lambda}}_1 \bar{\phi}_1 - \dot{\bar{\lambda}}_2 \bar{\phi}_2 - \bar{H}_c, \quad (74)$$

and new symplectic variables and corresponding one-form are

$$\begin{aligned} \bar{\xi}_\alpha^{(2)} &= (x, y, z, s, p_x, p_y, p_z, p_s, \bar{\lambda}_1, \bar{\lambda}_2), \\ \bar{\mathcal{A}}_\alpha^{(2)} &= (p_x, p_y, p_z, p_s, 0, 0, 0, 0, \bar{\phi}_1, \bar{\phi}_2). \end{aligned} \quad (75)$$

The corresponding symplectic two-form is

$$\bar{f}_{\alpha\beta}^{(2)} = \begin{pmatrix} \mathbf{0}_{4 \times 4} & -\mathbf{1}_{4 \times 4} & \mathcal{U}_{1 \times 4}^T & \mathcal{V}_{1 \times 4}^T \\ \mathbf{1}_{4 \times 4} & \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 1} & \mathcal{W}_{1 \times 4}^T \\ -\mathcal{U}_{1 \times 4} & \mathbf{0}_{1 \times 4} & 0 & 0 \\ -\mathcal{V}_{1 \times 4} & -\mathcal{W}_{1 \times 4} & 0 & 0 \end{pmatrix}, \quad (76)$$

in which, \mathcal{V}_α and \mathcal{W}_α are defined as fallow,

$$\mathcal{V}_\mu = \frac{\partial \bar{\phi}_2}{\partial \bar{q}^\mu}, \quad \mathcal{W}_\mu = \frac{\partial \bar{\phi}_2}{\partial \bar{p}^\mu}. \quad (77)$$

The two-form (76) is non-singular. Thus, it does not have any null vector and consequently there is no other constraint.

To start the symplectic embedding process, we expand the original Phase-space, using the unknown function depending on the phase-space variables and WZ variable, κ , which is defined as the following expansion with the same boundary condition as (26),

$$\bar{G}(x, y, z, s, p_x, p_y, p_z, p_s, \bar{\lambda}_1, \kappa) = \sum_{n=0}^{\infty} \mathfrak{g}^{(n)}, \quad (78)$$

Introducing the new term \bar{G} into the Lagrangian (66),

$$\check{L}^{(1)} = \bar{L}^{(1)} + \bar{G}(x, y, z, s, p_x, p_y, p_z, p_s, \bar{\lambda}_1, \kappa), \quad (79)$$

extends the symplectic variables as follows,

$$\begin{aligned}\check{\xi}_\alpha^{(1)} &= (x, y, z, s, p_x, p_y, p_z, p_s, \bar{\lambda}_1, \kappa), \\ \check{\mathcal{A}}_\alpha^{(1)} &= (p_x, p_y, p_z, p_s, 0, 0, 0, 0, \bar{\phi}_1, 0).\end{aligned}\quad (80)$$

Calculating the corresponding two-form symplectic matrix, we have

$$f_{\check{\alpha}\check{\beta}}^{(1)} = \begin{pmatrix} \bar{f}_{\alpha\beta}^{(1)} & \mathbf{0}_{9 \times 1} \\ \mathbf{0}_{1 \times 9} & 0 \end{pmatrix}, \quad (81)$$

which has the following null vectors,

$$\begin{aligned}\check{n}_{1\check{\alpha}}^{(1)} &= \begin{pmatrix} \bar{n}_{1\alpha}^{(1)} & 1 \end{pmatrix}, \\ \check{n}_{2\check{\alpha}}^{(1)} &= \begin{pmatrix} \bar{n}_{2\alpha}^{(1)} & 0 \end{pmatrix}.\end{aligned}\quad (82)$$

These null vectors also generate gauge transformations on the symplectic variables (80). To continue the procedure, we use the linear combination of them,

$$\check{n}_{\check{\alpha}} = \check{n}_{1\check{\alpha}}^{(1)} + \check{n}_{2\check{\alpha}}^{(1)} \quad (83)$$

Considering the fact that null vectors (82) terminates the constraint making process, we can use the following differential equation to obtain $\mathfrak{g}^{(n)}$,

$$\check{n}_{\check{\alpha}} \frac{\partial \bar{V}^{(1)}}{\partial \check{\xi}^{(0)\check{\alpha}}} = \frac{\partial \mathfrak{g}^{(n)}}{\partial \kappa}. \quad (84)$$

Substituting (14) into (84), we find $\mathfrak{g}^{(1)}$ as a linear function of WZ variable as,

$$\mathfrak{g}^{(1)} = \kappa \bar{\phi}_2. \quad (85)$$

Putting $\mathfrak{g}^{(1)}$ into (79), the potential will be

$$\check{V}^{(1)} = \bar{H}_c - \mathfrak{g}^{(1)}. \quad (86)$$

Using the (84) for the second time to get $\mathfrak{g}^{(2)}$, we will have

$$\mathfrak{g}^{(2)} = -\frac{\kappa^2}{2} \{\bar{\phi}_1, \bar{\phi}_2\}. \quad (87)$$

Substituting $\mathfrak{g}^{(2)}$ into the first iterative Lagrangian, we will obtain the second iterative Lagrangian with the following potential,

$$\check{V}^{(1)} = \bar{H}_c - \mathfrak{g}^{(1)} - \mathfrak{g}^{(2)}. \quad (88)$$

Again, using (84) to obtain $\mathfrak{g}^{(3)}$, we will see that $\frac{\partial \mathfrak{g}^{(3)}}{\partial \kappa} = 0$ and so, the zero-mode (83) does not make a new constraint. Thus, all correction terms $\mathfrak{g}^{(n)}$ with $n \geq 3$ are nulls. So, for the canonical Hamiltonian we have

$$\check{H}_{(c)} = \bar{H}_c + \bar{\lambda}_1 \bar{\phi}_1 - \mathfrak{g}^{(1)} - \mathfrak{g}^{(2)}, \quad (89)$$

and for the gauged Lagrangian,

$$\check{L}^{(1)} = \bar{L}^{(1)} + \mathfrak{g}^{(1)} + \mathfrak{g}^{(2)}. \quad (90)$$

As we mentioned before, in order to obtain gauge symmetries of the model, one can use (41) as an option [52, 53], or (39) and corresponding zero-modes (82) as another one [44, 49], which both give the same result.

$$\begin{aligned} \delta x &= 0, & \delta p_x &= \epsilon_1 \mathcal{U}_1, \\ \delta y &= 0, & \delta p_y &= \epsilon_1 \mathcal{U}_2, \\ \delta z &= 0, & \delta p_z &= \epsilon_1 \mathcal{U}_3, \\ \delta s &= 0, & \delta p_s &= \epsilon_1 \mathcal{U}_4, \\ \delta \bar{\lambda} &= \epsilon_2, & \delta \kappa &= \epsilon_1. \end{aligned} \quad (91)$$

These are nontrivial variations that make the system invariant under some gauge transformations. So, in the new model, there are some generators for gauge transformations which then we seek them.

3.3 Constraint structure of the gauged Lagrangian of a particle on 3-torus

Now, we can find the constraint structure of the gauged Lagrangian (90), using the same method as (42) and check consistency conditions. Thus, we have

$$\begin{aligned} \frac{\partial \check{L}^{(0)}}{\partial \lambda^{(1)}} &= 0 : \rightarrow \check{\rho}_1 = p_{\bar{\lambda}}, \\ \frac{\partial \check{L}^{(0)}}{\partial \dot{\kappa}^{(1)}} &= 0 : \rightarrow \check{\rho}_2 = p_{\kappa}. \end{aligned} \quad (92)$$

The constraint structure of the 3-torus is similar to the ordinary torus. Thus, we have the following chain structures.

$$\begin{aligned} \check{\rho}_1 &\rightarrow \check{\psi}_1 \rightarrow \check{\psi}_2 \rightarrow \times, \\ \check{\rho}_2 &\rightarrow \check{\psi}_2 \rightarrow \times. \end{aligned} \quad (93)$$

Here, $\check{\rho}_1$ is a first-class constraint, while its Poisson bracket and all constraints vanish. In order to make another first-order constraint we

should redefine them like (50),

$$\begin{aligned}\check{\Phi}_1^{(0)} &= p_{\bar{\lambda}}, \\ \check{\Phi}_1^{(1)} &= \check{\psi}_1, \\ \check{\Phi}_2^{(1)} &= \check{\psi}_2, \\ \check{\Phi}_3 &= \check{\rho}_2 + \check{\Phi}_1^{(1)},\end{aligned}\tag{94}$$

which $\check{\Phi}_1^{(0)}$ and $\check{\Phi}_3$ are first-class, and $\check{\Phi}_1^{(1)}$ and $\check{\Phi}_2^{(1)}$ are second-class constraints.

Now, we can make the canonical Hamiltonian,

$$\check{H}_{(c)} = \bar{H}_c + \bar{\lambda}_1 \check{\Phi}_1^{(1)} + \kappa \check{\Phi}_2^{(1)}.\tag{95}$$

Similar to the torus model, added coordinates to the extended phase-space have the following dimensions,

$$[\bar{\lambda}] = (Length)^{-4}, \quad [\kappa] = (Length)^{-2}.$$

Then, we rewrite the gauged canonical Hamiltonian, using the variables with length dimension as,

$$\check{H}_c = \bar{H}_c + \frac{1}{\lambda'^4} \check{\Phi}_1^{(1)} + \frac{1}{\kappa'^2} \check{\Phi}_2^{(1)}.\tag{96}$$

As same as torus model, λ' and κ' can be interpreted as two large extra dimensions which are added to the phase-space.

We see that the constrained structure for a free particle on the 3-torus is similar to the torus. Thus, its Poisson structure and Dirac brackets are somehow similar. One can obtain non-vanishing Dirac brackets of this model, using (55). Also, we can check the effect of the 3-torus topology on Dirac brackets by expanding the Poisson brackets of the gauged model with respect to the ratios of radiuses of the torus, considering both $\check{\zeta}_1 \rightarrow 0$ and $\check{\zeta}_2 \rightarrow 0$.

Like the torus model, we see that these Dirac brackets do not have the common canonical structure (See B). Thus, we can interpret such deformations from canonical structure as the effect of topology on the Poisson structure of particle living on a 3-torus.

3.4 How to extract phenomenology related to MOND theory

As we know, to find Poisson structure between two functions A and B in a phase-space which owns itself a Poisson structure as $\{\xi_\alpha, \xi_\beta\}^*$, one

can use the following relation [54, 55].

$$\{A, B\} = \{\xi_\alpha, \xi_\beta\}^* \left(\frac{\partial A}{\partial \xi_\alpha} \frac{\partial B}{\partial \xi_\beta} \right), \quad (97)$$

which $\{\xi_\alpha, \xi_\beta\}^*$ is the Dirac bracket between phase-space variables. The relation (97) is used to obtain the time evolution of phase-space variables and consequently the Hamilton equations of motion.

In order to relate the obtained gauge theory on the 3-torus and MOND theory [56, 57, 58, 59], we rewrite the Hamilton equations of motion for dynamical variables of our gauged model and combine them to obtain the Newton's second law, and more interestingly its corresponding corrections.

To start with, we consider the general radial potential $V(r)$, added to the Hamiltonian which does not change the derived gauged theory.

$$H = \frac{p_{i'} p^{i'}}{2} + V(r), \quad (98)$$

which r is the radial vector, i.e. $r^2 = x^2 + y^2 + z^2 + s^2$, in the Cartesian coordinates.

Hence, due to the nontrivial Poisson structure, the Newton's second law in the direction of $x_{i'}$ will be

$$\ddot{x}_{i'} = -\frac{\partial V}{\partial x_{i'}} + \mathcal{F}_{i'}(\xi_\alpha, \dot{\xi}_\beta), \quad (99)$$

where, $\mathcal{F}_{i'}(\xi_\alpha, \dot{\xi}_\beta)$ can be interpreted as the modification term of the Newton's second law.

Let's start studying this modification in details. As we mentioned before, due to the amendments which are imposed by Dirac brackets ascribed to second-class constraints, the equations of motions shall surely include some corrections. Thus, for the equations of motions we have,

$$\begin{aligned} \dot{x}_{i'} &\approx \{x_{i'}, H_c\} = \{x_{i'}, \xi_\alpha\} \frac{\partial H}{\partial \xi_\alpha}, \\ \ddot{x}_{i'} &\approx \{\{x_{i'}, H_c\}, H_c\}. \end{aligned} \quad (100)$$

Calculating the explicit relation of acceleration, i.e. $\ddot{x}_{i'}$, we arrive to

$$\ddot{x}_{i'} \approx \{x_{i'}, p_i\} \left\{ \frac{\partial H_c}{\partial p_i}, H_c \right\} + \frac{\partial H_c}{\partial \xi_\alpha} \left(\{\{x_{i'}, p_j\} \frac{\partial H_c}{\partial p_j}, \xi_\alpha\} - \{\{\xi_\alpha, H_c\}, x_{i'}\} \right). \quad (101)$$

It is easy to investigate that the equations of motion in the constructed theory give the following correction for the acceleration of the

free particle which does not feel any potential. As we know, these corrections are equal to zero for common situations.

$$\ddot{x}_{i'} \approx \dot{x}_{i'} \left[-\frac{p_i p^i}{r^2} + 5 \frac{(x_i p^i)^2}{r^4} \right] - 4 p_{i'} \left[\frac{x_i p^i}{r^2} \right]. \quad (102)$$

It is evident that these accelerations depend on the position, momentum and the kinetic energy of the particle and are of the order of $(Length)^{-1}$.

For a particle with unit mass which is affected by the potential in a theory with flat space and the topology \mathbb{R}^3 , all the terms in (99) vanish but the first term $-\frac{\partial V}{\partial x_{i'}}$. So, we obtain the Newton's second law as $\mathcal{F} = \ddot{x}_{i'}$.

On the other hand, for a gauged theory, the relation (102) contains extra terms which have the capability to be explained as the correction of the Newton's second law. These correction terms can be used for phenomenology of the MOND theory, and somehow, be the candidate to explain dark matter.

As a matter of fact, these corrections are classified into two categories. First, terms which are added to (100) in a proper gauge, to convert weak equalities to strong ones, or in other words, changing H_c to H_T , in order to gain full dynamics of the particle. Moreover, for the case of having the common Poisson structure, the only survived factor which is derived from the first term of (99), is $-\frac{\partial V}{\partial x_{i'}}$. But, in the model of the particle, living on the torus, as we saw in the last section, we encountered some deviations in the common Poisson structure of the phase-space. Thus, aside from the survived first term which adds some corrections itself because of the deformed Poisson structure of the phase-space, there are also correction terms, arised from other terms. All these modification factors can be used to study MOND.

In addition, if we consider the gravitational potential of the mass M as $V = \frac{GM}{r^2}$, then we can find the correction which is imposed on the universal law of gravity via the attendance of the particle on the torus in tree-dimensional space, and in the presence of the extra gauged coordinates [60, 61, 62]. Thus, one can conclude that for the particle which is affected by the gravitational potential, despite the added corrections for accelerations, other corrections are emerged for the gravitational potential itself.

Moreover, these modification factors can be interpreted as the functions of the new coordinates λ' and κ' . From this point of view, we can elucidate these terms as large extra dimensions in Randall-Sundrum scenario [19, 63].

Conclusion

Based on the observations and the theories in cosmology, we assume that our universe has a topology of a torus. We consider a non-relativistic particle as a test object in this background and find the corresponding gauged phase-space in classical mechanics framework to obtain classical equations of the particle. Also, using Dirac's approach, we gain the particle's quantum mechanics to the first order of \hbar and associated Hilbert space.

First, we construct a gauge theory on a torus in three-dimensional space as a toy model. We try to concede two degrees of freedom to this particle which lives on a two-dimensional world, using symplectic embedding approach. We show that the interaction which is added to the Hamiltonian of the particle via this process is as the inverse of the length to the power of four on the axis of the gauged degrees of freedom. We also show that these two added degrees of freedom is due to the imposing the constraint of being on the torus on the Hamiltonian with the help of Lagrange multipliers, and the gauging WZ variables. Extracting its Poisson structure, we can quantize the model, by replacing Dirac brackets with quantum commutators. Afterwards, we study the effect of the topology on Dirac brackets by enlarging one radius of the torus in comparison with another. We witness that the model of a free particle in the torus converts to a particle on an infinite ring with the topology of S^1 , which is the very effect of the topology on the Poisson structure of the phase-space.

With the help of this toy model, in the main part of this article, we constrain the particle on the hyper-surface of a three-torus in a four dimensional configuration space. Because of the second-class constraints which are imposed on such a particle, the particle will live on a six-dimensional phase-space or three-dimensional configuration space. Then, using the symplectic formalism and embedding the phase-space to an extended one, to the particle's motion degrees of freedom we add two more ones. These degrees of freedom are equivalent to two second-class constraints, gauged in the final model. By constructing the main brackets of the phase-space, we obtain classic and quantum mechanics of the particle, from which we can interpret deviations occurred from classic and quantum mechanics in \mathbb{R}^3 space due to the topology of the torus. Hence, we can say that the effect of the topology $S^1 \times S^1 \times S^1$ on Dirac brackets, leads to the presence of the particle on a three dimensional manifold, i.e. the particle travels on the surface of the 3-torus and is not floated in its bulk. The obtained quantum mechanics can be investigated by studying the energy spectrum of the free particle in the

model. In the gauged particle's quantum mechanics which is affected by the space topology, main commutation relations change basically. So, even in the free particle's Schrodinger equation of motion, there will be an added potential which makes the corresponding energy spectrum discrete. Hence, by investigating the free particle's energy spectrum and comparing with the very spectrum which is obtained via quantum mechanics of the model, we can determine the corresponding parameters accurately.

Another research work in cosmology can be done with the help of above model to obtain the proposed corrections from MOND theory. Using the equations of motion obtained from the gauged Hamiltonian, we gain the modified form of the Newton's second law. The obtained correction terms in our model can be compared with the cosmological observations for investigating MOND, and at the same time, it verifies the MOND theoretically.

Moreover, the obtained model has two extra gauged degrees of freedom comparing to the common three-dimensional space's degrees of freedom, which add interactions proportion to inverse length square and inverse of the length to the power of four to Hamiltonian. Particularly, if we consider that the test particle is affected by the gravitational potential of another one in the origin of the space, then in the gauged model, the particle faces the corrections due to the two extra dimensions, which can be related to the brane world cosmology.

A

As we mentioned before, we reduce the primary constraint of a particle model on a 3-torus to a polynomial by eliminating the radicals as follow,

$$\begin{aligned}
\bar{\phi}_1 = & x^8 + y^8 + z^8 + s^8 + 4x^2s^6 + 4y^2s^6 + 4z^2s^6 - 4\zeta_1^2s^6 - 4\zeta_2^2s^6 - 4s^6 + 6x^4s^4 \\
& + 6y^4s^4 + 6z^4s^4 + 6\zeta_1^4s^4 + 6\zeta_2^4s^4 - 12x^2s^4 + 12x^2y^2s^4 - 12y^2s^4 + 12x^2z^2s^4 \\
& + 12y^2z^2s^4 - 12z^2s^4 - 4x^2\zeta_1^2s^4 - 12y^2\zeta_1^2s^4 - 12z^2\zeta_1^2s^4 + 4\zeta_1^4s^4 - 4x^2\zeta_2^2s^4 \\
& - 4y^2\zeta_2^2s^4 - 12z^2\zeta_2^2s^4 - 60\zeta_1^2\zeta_2^2s^4 + 4\zeta_2^2s^4 + 6s^4 + 4x^6s^2 + 4y^6s^2 + 4z^6s^2 \\
& - 4\zeta_1^6s^2 - 4\zeta_2^6s^2 - 12x^4s^2 + 12x^2y^4s^2 - 12y^4s^2 + 12x^2z^4s^2 + 12y^2z^4s^2 - 12z^4s^2 \\
& - 4x^2\zeta_1^4s^2 + 12y^2\zeta_1^4s^2 + 12z^2\zeta_1^4s^2 + 4\zeta_1^4s^2 - 4x^2\zeta_2^4s^2 - 4y^2\zeta_2^4s^2 + 12z^2\zeta_2^4s^2 \\
& - 60\zeta_1^2\zeta_2^4s^2 + 4\zeta_2^4s^2 + 12x^2s^2 + 12x^4y^2s^2 - 24x^2y^2s^2 + 12y^2s^2 + 12x^4z^2s^2 \\
& + 12y^4z^2s^2 - 24x^2z^2s^2 + 24x^2y^2z^2s^2 - 24y^2z^2s^2 + 12z^2s^2 + 4x^4\zeta_1^2s^2 - 12y^4\zeta_1^2s^2 \\
& - 12z^4\zeta_1^2s^2 - 8x^2\zeta_1^2s^2 - 8x^2y^2\zeta_1^2s^2 + 8y^2\zeta_1^2s^2 - 8x^2z^2\zeta_1^2s^2 - 24y^2z^2\zeta_1^2s^2 \\
& + 8z^2\zeta_1^2s^2 + 4\zeta_1^2s^2 + 4x^4\zeta_2^2s^2 + 4y^4\zeta_2^2s^2 - 12z^4\zeta_2^2s^2 - 60\zeta_1^4\zeta_2^2s^2 - 8x^2\zeta_2^2s^2 \\
& + 8x^2y^2\zeta_2^2s^2 - 8y^2\zeta_2^2s^2 - 8x^2z^2\zeta_2^2s^2 - 8y^2z^2\zeta_2^2s^2 + 8z^2\zeta_2^2s^2 - 24x^2\zeta_1^2\zeta_2^2s^2 \\
& - 72y^2\zeta_1^2\zeta_2^2s^2 - 120z^2\zeta_1^2\zeta_2^2s^2 + 24\zeta_1^2\zeta_2^2s^2 + 4\zeta_2^2s^2 - 4s^2 - 4x^6 - 4y^6 - 4z^6 + 4x^2y^6 \\
& + 4x^2z^6 + 4y^2z^6 - 4x^2\zeta_1^6 - 4y^2\zeta_1^6 - 4z^2\zeta_1^6 - 4\zeta_1^6 + 4x^2\zeta_2^6 + 4y^2\zeta_2^6 - 4z^2\zeta_2^6 \\
& - 4\zeta_2^6 - 4\zeta_2^6 + 6x^4 + 6x^4y^4 - 12x^2y^4 + 6y^4 + 6x^4z^4 + 6y^4z^4 - 12x^2z^4 \\
& + 12x^2y^2z^4 - 12y^2z^4 + 6z^4 + 6x^4\zeta_1^4 + 6y^4\zeta_1^4 + 6z^4\zeta_1^4 - 12x^2\zeta_1^4 - 4x^2y^2\zeta_1^4 \\
& + 4y^2\zeta_1^4 - 4x^2z^2\zeta_1^4 + 12y^2z^2\zeta_1^4 + 4z^2\zeta_1^4 + 6\zeta_1^4 + 6x^4\zeta_2^4 + 6y^4\zeta_2^4 + 6z^4\zeta_2^4 + 6\zeta_1^4\zeta_2^4 \\
& - 12x^2\zeta_2^4 + 12x^2y^2\zeta_2^4 - 12y^2\zeta_2^4 - 4x^2z^2\zeta_2^4 - 4y^2z^2\zeta_2^4 + 4z^2\zeta_2^4 - 4x^2\zeta_1^2\zeta_2^4 \\
& - 12y^2\zeta_1^2\zeta_2^4 - 60z^2\zeta_1^2\zeta_2^4 + 4\zeta_1^2\zeta_2^4 + 6\zeta_2^4 - 4x^2 + 4x^6y^2 - 12x^4y^2 + 12x^2y^2 - 4y^2 \\
& + 4x^6z^2 + 4y^6z^2 - 12x^4z^2 + 12x^2y^4z^2 - 12y^4z^2 + 12x^2z^2 + 12x^4y^2z^2 \\
& - 24x^2y^2z^2 + 12y^2z^2 - 4z^2 + 4x^6\zeta_1^2 - 4y^6\zeta_1^2 - 4z^6\zeta_1^2 - 12x^4\zeta_1^2 - 4x^2y^4\zeta_1^2 \\
& + 4y^4\zeta_1^2 - 4x^2z^4\zeta_1^2 - 12y^2z^4\zeta_1^2 + 4z^4\zeta_1^2 + 12x^2\zeta_1^2 + 4x^4y^2\zeta_1^2 - 8x^2y^2\zeta_1^2 \\
& + 4y^2\zeta_1^2 + 4x^4z^2\zeta_1^2 - 12y^4z^2\zeta_1^2 - 8x^2z^2\zeta_1^2 - 8x^2y^2z^2\zeta_1^2 + 8y^2z^2\zeta_1^2 + 4z^2\zeta_1^2 - 4\zeta_1^2 \\
& + 4x^6\zeta_2^2 + 4y^6\zeta_2^2 - 4z^6\zeta_2^2 - 4\zeta_1^6\zeta_2^2 - 12x^4\zeta_2^2 + 12x^2y^4\zeta_2^2 - 12y^4\zeta_2^2 - 4x^2z^4\zeta_2^2 \\
& - 4y^2z^4\zeta_2^2 + 4z^4\zeta_2^2 - 4x^2\zeta_1^4\zeta_2^2 + 12y^2\zeta_1^4\zeta_2^2 - 60z^2\zeta_1^4\zeta_2^2 + 4\zeta_1^4\zeta_2^2 + 12x^2\zeta_2^2 \\
& + 12x^4y^2\zeta_2^2 - 24x^2y^2\zeta_2^2 + 12y^2\zeta_2^2 + 4x^4z^2\zeta_2^2 + 4y^4z^2\zeta_2^2 - 8x^2z^2\zeta_2^2 + 8x^2y^2z^2\zeta_2^2 \\
& - 8y^2z^2\zeta_2^2 + 4z^2\zeta_2^2 + 4x^4\zeta_1^2\zeta_2^2 - 12y^4\zeta_1^2\zeta_2^2 - 60z^4\zeta_1^2\zeta_2^2 - 8x^2\zeta_1^2\zeta_2^2 - 8x^2y^2\zeta_1^2\zeta_2^2 \\
& + 8y^2\zeta_1^2\zeta_2^2 - 24x^2z^2\zeta_1^2\zeta_2^2 - 72y^2z^2\zeta_1^2\zeta_2^2 + 24z^2\zeta_1^2\zeta_2^2 + 4\zeta_1^2\zeta_2^2 - 4\zeta_2^2 + \zeta_1^8 + \zeta_2^8 + 1
\end{aligned}$$

B

Dirac brackets of the phase-space variables are shown in the following table. As it is evident, these brackets do not include any correction to the first order of $\check{\zeta}_i$, i.e. $\{\check{\zeta}_\alpha, \check{\zeta}_\beta\} \simeq J_{\alpha\beta} + O(\check{\zeta}_1^2) + O(\check{\zeta}_2^2) + O(\check{\zeta}_1\check{\zeta}_2)$. Here, we define $r^2 = x^2 + y^2 + z^2 + s^2$.

$\{, \}^*$	q_j	λ	κ	p_j	p_λ	p_κ
q_i	0	0	0	$\delta_{ij} - \frac{q_i q_j}{r^2}$	0	0
λ		0	0	0	1	0
κ			0	0	0	1
p_i				$\frac{L_{ij}}{r^2}$	0	$8q_i(r^2 - 1)^3$
p_λ					0	0
p_κ						0

Table 2: Dirac brackets between extended phase-space variables of the 3-torus model

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